

Generalizations of a method for constructing first integrals of a class of natural Hamiltonians and some remarks about quantization

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Abstract

In previous papers we determined necessary and sufficient conditions for the existence of a class of natural Hamiltonians with non-trivial first integrals of arbitrarily high degree in the momenta. Such Hamiltonians were characterized as $(n+1)$ -dimensional extensions of n -dimensional Hamiltonians on constant-curvature (pseudo-)Riemannian manifolds Q . In this paper, we generalize that approach in various directions, we obtain an explicit expression for the first integrals, holding on the more general case of Hamiltonians on Poisson manifolds, and show how the construction of above is made possible by the existence on Q of particular conformal Killing tensors or, equivalently, particular conformal master symmetries of the geodesic equations. Finally, we consider the problem of Laplace-Beltrami quantization of these first integrals when they are of second-degree.

1 Introduction

In recent years, several progresses have been done in the field of integrable and superintegrable Hamiltonian systems, both classical and quantum, by the introduction of new techniques for the study of higher-degree polynomial first integrals and higher-order symmetry operators. After researches exposed in [4],

[7] and [10] is now possible to explicitly build and analyze Hamiltonian systems possessing symmetries of arbitrarily-high degree. For a more detailed introduction see the contribution to the QTS 7 proceedings written by W. Miller Jr. In several papers ([4], [5], [6]) we developed the analysis of a class of systems which, in dimension two, are a subset of the celebrated Tremblay-Turbiner-Winternitz (TTW) systems and are strictly related with the Jacobi-Calogero and Wolfes three-body systems [4], [6]. In [5] we generalized these systems to higher-dimensions by introducing a $(n+1)$ -dimensional extension H of a given n -dimensional natural Hamiltonian L . We obtained necessary and sufficient conditions for the existence of a first integral of H in a particular form, one necessary condition being the constant curvature of the configuration manifold on which L is defined (for superintegrable systems with higher-degree first integrals on constant curvature manifolds see also [8]). The first integral of H , which is independent from those of L , is polynomial in the momenta and can be explicitly constructed through a differential operator. In the present paper, we generalize the analysis done in [5] in several directions. In Sec. 2 we extend the construction to non natural Hamiltonians on a general Poisson manifolds and obtain, also in this case, an explicit expression for the polynomial first integral. In Sec. 3 we restrict ourselves to cotangent bundles of (pseudo-)Riemannian manifolds and consider a wider class of higher-degree first integrals, we prove that a necessary condition for their existence is the presence of a particular class of conformal Killing tensors or, equivalently, of conformal master symmetries of the geodesic equations; we end the section with an example showing how the method can provide several independent first integrals of degree m . In Sec. 4 we characterize our construction in an invariant way and determine necessary and sufficient conditions for the constant curvature or conformal flatness of the configuration manifold of H , conditions employed in Sec. 5, where the quantization of the second-degree first integrals obtained by our method is considered.

2 Extensions on a Poisson manifold

Let us consider a Poisson manifold M and a one-dimensional manifold N . For any Hamiltonian function $L \in \mathcal{F}(M)$ with Hamiltonian vector field X_L , we consider its extension on $\tilde{M} = T^*N \times M$ given by the Hamiltonian

$$H = \frac{1}{2}p_u^2 + \alpha(u)L + \beta(u) \quad (1)$$

where (p_u, u) are canonical coordinates on T^*N and $\alpha(u) \neq 0$. The Hamiltonian flow of (1) is

$$X_H = p_u \frac{\partial}{\partial u} - (\dot{\alpha}L + \dot{\beta}) \frac{\partial}{\partial p_u} + \alpha X_L,$$

where dots denotes total derivative w.r.t. the (single) variable u .

It is immediate to see that any first integral of L is also a constant of motion of H , when considered as a function on \tilde{M} . We recall that a function F is a first integral of H if and only if $X_H F = \{H, F\} = 0$.

In [5] we determined on L , α and β necessary and sufficient conditions for the existence of two functions $\gamma \in \mathcal{F}(N)$ and $G \in \mathcal{F}(M)$ such that, given the differential operator

$$U = p_u + \gamma(u)X_L, \quad (2)$$

the function $U^m(G)$ obtained applying $m \neq 0$ times U to G is a non trivial additional first integral for H .

In particular, if L is a natural Hamiltonian on the cotangent bundle of a (pseudo-)Riemannian manifold (Q, \mathbf{g})

$$L = \frac{1}{2}g^{ij}p_i p_j + V$$

and α is assumed to be not constant, an integral of the form $U^m(G)$ exists, with G not dependent on the momenta, if and only if G satisfy for some constant $c \neq 0$ the equations:

$$\nabla_i \nabla_j G + mcg_{ij}G = 0, \quad (3)$$

$$\nabla^i V \nabla_i G = 2mcVG, \quad (4)$$

which are equivalent to

$$\{\nabla^i G p_i, L\} = 2mcGL,$$

meaning that $\nabla^i G p_i$ is a conformal first integral of L .

If a solution of the previous equations exists, then the extended Hamiltonian (1) and the differential operator (2) take the form

$$H = \frac{1}{2}p_u^2 + \frac{mc}{S_\kappa^2(cu + u_0)}L \quad (5)$$

$$U = p_u + \frac{1}{T_\kappa(cu + u_0)}X_L, \quad (6)$$

where the trigonometric tagged functions (see [3, 9]) are employed

$$S_\kappa(x) = \begin{cases} \frac{\sin \sqrt{\kappa}x}{\sqrt{\kappa}} & \kappa > 0 \\ x & \kappa = 0 \\ \frac{\sinh \sqrt{|\kappa}|x}{\sqrt{|\kappa|}} & \kappa < 0 \end{cases} \quad T_\kappa(x) = \begin{cases} \frac{\tan \sqrt{\kappa}x}{\sqrt{\kappa}} & \kappa > 0 \\ x & \kappa = 0 \\ \frac{\tanh \sqrt{|\kappa}|x}{\sqrt{|\kappa|}} & \kappa < 0 \end{cases}$$

Here we show that an analogous result holds in a general situation.

Proposition 1. *Let H be the extension (1) of the Hamiltonian L on the Poisson manifold \tilde{M} , let U the differential operator (2) and $G \in \mathcal{F}(M)$ a function such that $X_L(G) \neq 0$. Then, $U^m(G)$ is a first integral for H if and only if G satisfies*

$$X_L^2(G) + 2m(cL + L_0)G = 0 \quad c, L_0 \in \mathbb{R}. \quad (7)$$

and α, β and γ satisfy

$$\alpha = -m\dot{\gamma}, \quad (8)$$

$$\beta = mL_0\gamma^2 + \beta_0, \quad \beta_0 \in \mathbb{R}, \quad (9)$$

$$\ddot{\gamma} + 2c\gamma\dot{\gamma} = 0. \quad (10)$$

Proof. In [5] it is proved that we have that $X_H U^m(G) = 0$ for a function $G \in \mathcal{F}(M)$ if and only if L, α, β satisfy

$$(m\dot{\gamma} + \alpha)X_L(G) = 0, \quad (11)$$

$$\alpha\gamma X_L^2(G) - m(\dot{\alpha}L + \dot{\beta})G = 0. \quad (12)$$

Because $X_L(G) \neq 0$, from (11) it follows that

$$\alpha = -m\dot{\gamma} \quad (13)$$

and condition (12) becomes

$$\dot{\gamma}\gamma \frac{X_L^2(G)}{G} = m\ddot{\gamma}L - \dot{\beta}.$$

Since $\dot{\gamma} = -\alpha/m \neq 0$, we get

$$\frac{X_L^2(G)}{G} = m \frac{\ddot{\gamma}}{\gamma\dot{\gamma}} L - \frac{\dot{\beta}}{\gamma\dot{\gamma}}, \quad (14)$$

which derived with respect to u gives

$$\frac{d}{du} \left(\frac{\ddot{\gamma}}{\gamma\dot{\gamma}} \right) L = \frac{d}{du} \left(\frac{\dot{\beta}}{m\gamma\dot{\gamma}} \right).$$

But L is a non-constant function on M , hence the functions $\ddot{\gamma}$ and $\dot{\beta}$ must be both proportional to $\gamma\dot{\gamma}$:

$$\begin{aligned} \ddot{\gamma} &= -2c\gamma\dot{\gamma} = -c \frac{d}{du} (\gamma^2), \\ \dot{\beta} &= 2mL_0\gamma\dot{\gamma} = mL_0 \frac{d}{du} (\gamma^2). \end{aligned}$$

By integrating and substituting in (14), we obtain conditions (7) and (9). \square

Remark 1. If $X_L(G) = 0$ we trivially have $U^m(G) = p_u^m$, which is a first integral of H only if α and β are constant. Hence, it is a constant of motion functionally dependent on L and H .

Remark 2. The equation (7) is obviously equivalent to

$$\{L, \{L, G\}\} = -2m(cL + L_0)G;$$

this condition can be interpreted in terms of master symmetries: the Hamiltonian vector field X_G is a master symmetry for the Hamiltonian vector field X_L on the hypersurfaces $L = 0$ or $G = 0$. Further remarks about the special case when L is a natural Hamiltonian are at the end of Sec. 3.

By integrating the equations for α , β and γ in Proposition 1 the explicit expression for the extended Hamiltonian H and the differential operator U can be found. From equation (7) we have [5]

Theorem 2. *Let H be the extension (1) of the Hamiltonian L on the Poisson manifold \tilde{M} , let U the differential operator (2) and $G \in \mathcal{F}(M)$ a function satisfying $X_L(G) \neq 0$ and (7). Then, $U^m G$ is a first integral of H if and only if H and U are in either one of the two following forms characterized by the value of c in (7)*

i) for $c \neq 0$

$$\begin{aligned} H &= \frac{1}{2}p_u^2 + \frac{mc}{S_\kappa^2(cu + u_0)}(L + V_0) + W_0, \\ U &= p_u + \frac{1}{T_\kappa(cu + u_0)}X_L, \end{aligned} \quad (15)$$

ii) for $c = 0$

$$\begin{aligned} H &= \frac{1}{2}p_u^2 + mA(L + V_0) + B(u + u_0)^2, \\ U &= p_u - A(u + u_0)X_L, \end{aligned} \quad (16)$$

with $\kappa, V_0, W_0 \in \mathbb{R}$, $B = mL_0A^2$ and $A \neq 0$.

Proof. By Proposition 1, α, β, γ must satisfy (8), (9), (10). In the case $c \neq 0$ equation (10) becomes $\dot{\gamma} + c(\gamma^2 + \kappa) = 0$, whose solution is

$$\gamma = \frac{1}{T_\kappa(cu + u_0)}.$$

Hence,

$$\begin{aligned} \alpha &= \frac{mc}{S_\kappa^2(cu + u_0)}, \\ \beta &= \frac{mcV_0}{S_\kappa^2(cu + u_0)} + W_0, \end{aligned}$$

with $V_0 = L_0/c$ and $W_0 = \beta_0 - m\kappa L_0$. In the case $c = 0$, equation (10) gives $\dot{\gamma} + A = 0$ with $A \neq 0$ in order to avoid $\alpha = 0$. Hence,

$$\begin{aligned} \alpha &= mA, \\ \beta &= mAV_0 + B(u + u_0)^2, \\ \gamma &= -A(u + u_0), \end{aligned}$$

where V_0 is now an arbitrary constant. \square

Remark 3. The constants u_0, V_0 and W_0 are not essential. Indeed, H and L are defined up to additive constant W_0 and V_0 while u_0 can be eliminated by a translation of u . In the case $c \neq 0$, the choice $V_0 = W_0 = 0$ gives the expressions (5) and (6) for H and U obtained in [5]. Moreover, by including the constant L_0 in the Hamiltonian L , the condition (7) assumes the simpler form

$$X_L^2(G) + 2mcLG = 0.$$

Once L and G satisfy condition (7) the first integrals $U^m(G)$ are explicitly determined for any $G : M \rightarrow \mathbb{R}$.

Theorem 3. Under the hypothesis of Proposition 1 the functions U^mG can be explicitly written as

$$U^mG = P_mG + D_mX_LG, \quad (17)$$

where

$$P_m = \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m}{2k} \gamma^{2k} p_u^{m-2k} (-2m(cL + L_0))^k,$$

$$D_m = \sum_{k=0}^{\lfloor m/2 \rfloor - 1} \binom{m}{2k+1} \gamma^{2k+1} p_u^{m-2k-1} (-2m(cL + L_0))^k, \quad m > 1,$$

where $\lfloor \cdot \rfloor$ denotes the integer part and $D_1 = \gamma$.

Proof. From equation (7) it follows that for all $k \in \mathbb{N}$ we have

$$X_L^{2k+1}G = (-2m(cL + L_0))^k X_L G, \quad X_L^{2k}G = (-2m(cL + L_0))^k G. \quad (18)$$

By expanding U^m using the binomial formula

$$U^m G = (p_u + \gamma X_L)^m = \sum_{k=0}^m \binom{m}{k} p_u^k (\gamma X_L)^{m-k},$$

and separating even and odd terms in k , by taking in account relations (18) we get equation (17). \square

The setting described in the previous section can be further generalized as follows. Let X_L be a Hamiltonian vector field on a Poisson manifold \tilde{M} , let on \tilde{M}

$$X_H = Y + f_3 X_L,$$

for a vector field Y and

$$U = f_1 + f_2 X_L,$$

where $f_i : \tilde{M} \rightarrow \mathbb{R}$. Following the same proof procedure as in [5] we get

Proposition 4. *If $X_L(f_i) = 0$ and $[Y, X_L] = 0$ then $X_H U^m(G) = 0$, i.e. $U^m(G)$ is a first integral of H , if and only if*

$$(f_1 Y + (mY(f_2) + f_1 f_3)X_L + f_2 X_L Y + f_2 f_3 X_L^2)(G) = -mY(f_1)G. \quad (19)$$

Proof. If $X_L(f_i) = 0$ and $[Y, X_L] = 0$, then

$$\{H, L\} = 0,$$

$$[X_H, U] = Y(f_1) + Y(f_2)X_L,$$

$$[[X_H, U], U] = 0.$$

Thus,

$$\begin{aligned} X_H U^m &= U^{m-1}(m[X_H, U] + U X_H) = \\ &= U^{m-1}(mY(f_1) + f_1 Y + (mY(f_2) + f_1 f_3)X_L + f_2 X_L Y + f_2 f_3 X_L^2). \end{aligned}$$

and the thesis follows. \square

The analysis of such a generalization will not be considered here.

3 Extensions of a natural Hamiltonian

In the following sections we will assume that L is a natural n -dimensional Hamiltonian on $M = T^*Q$ for a (pseudo-)riemannian manifold (Q, \mathbf{g}) :

$$L = \frac{1}{2}g^{ij}(q^h)p_i p_j + V(q^h), \quad (20)$$

where g^{ij} are the contravariant components of the metric tensor and V a scalar potential. This assumption, together with the hypothesis that G is polynomial of degree d in the momenta (p_i) , allows us to expand condition (7) into an equality of two polynomials in (p_i) of degree $d+2$ that can be splitted into several differential conditions involving the metric, the potential and the coefficients of G . Indeed, being L a natural Hamiltonian, we have (in [5] the equation for X_L^2 was mistyped, however, this does not affects any of the results of the paper,

$$X_L = p_i \nabla^i - \nabla_i V \frac{\partial}{\partial p_i},$$

$$X_L^2 = p_i p_j \nabla^i \nabla^j - \nabla_i V \nabla^i - 2p_j \nabla_i V \nabla^j \frac{\partial}{\partial p_i} - p_i \nabla^i \nabla_j V \frac{\partial}{\partial p_j} + \nabla_i V \nabla_j V \frac{\partial^2}{\partial p_i \partial p_j}.$$

In [5] we dealt with the case $c \neq 0$, $d = 0$, i.e. G independent of momenta, obtaining the conditions (3) and (4). The maximal dimension of the space of solutions of equation (3) is $n + 1$ and it is achieved only if the metric \mathbf{g} on Q has constant curvature. We call *complete* the solutions G of (3) satisfying this integrability condition (see [5]).

In the following, we analyze in details the $d = 1$ case (G linear in the momenta), in order to show how the procedure works.

Proposition 5. *Let be $G = \lambda^l(q^i)p_l + W(q^i)$. Then, $U^m G$ is a first integral of H if and only if*

$$\nabla^i \nabla^j \lambda^l + mcg^{ij} \lambda^l = 0, \quad (21)$$

$$\nabla^i \nabla^j W + mcg^{ij} W = 0, \quad (22)$$

$$\nabla_i V (\nabla^i \lambda^l + 2\nabla^l \lambda^i) + \lambda^i \nabla^l \nabla_i V - 2m\lambda^l (cV + L_0) = 0, \quad (23)$$

$$\nabla_i V \nabla^i W - 2m(cV + L_0)W = 0, \quad (24)$$

Proof. For G linear in the momenta we have

$$\begin{aligned} X_L G &= p_i p_l \nabla^i \lambda^l - \nabla_i V \lambda^i + p_i \nabla^i W, \\ X_L^2 G &= p_i p_j p_l \nabla^i \nabla^j \lambda^l - p_l (\nabla_i V (\nabla^i \lambda^l + 2\nabla^l \lambda^i) + \lambda^i \nabla^l \nabla_i V) + \\ &\quad + p_i p_j \nabla^i \nabla^j W - \nabla_i V \nabla^i W, \end{aligned}$$

and condition (7) holds if and only if

$$\begin{aligned} p_i p_j p_l (\nabla^i \nabla^j \lambda^l + mcg^{ij} \lambda^l) + p_i p_j (\nabla^i \nabla^j W + mcg^{ij} W) - \\ p_l (\nabla_i V (\nabla^i \lambda^l + 2\nabla^l \lambda^i) + \lambda^i \nabla^l \nabla_i V - 2m\lambda^l (cV + L_0)) + \\ 2m(cV + L_0)W - \nabla_i V \nabla^i W = 0, \end{aligned}$$

which is equivalent to eqs. (21, 22, 23, 24). \square

Remark 4. The coefficients of terms with even and odd degree in the momenta are involved in different equations: eq.s (22) and (24) contain the ones of a G independent of p_i . Hence, for $\lambda^i = 0$ we recover the $d = 0$ case: (22) and (24) are the expansion in coordinates of (3) and (4) for $G = W$. For $W \neq 0$ the compatible potentials V have to satisfy both conditions (24) and (23), thus it is impossible to get new potentials other than those compatible with a G independent of the momenta i.e., satisfying conditions (22–24).

From (22) one can derive (see [5]) integrability conditions for W

$$(R_{hijk} - mc(g_{hj}g_{ik} - g_{hk}g_{ij})) \nabla^h \ln W = 0. \quad (25)$$

If these equations are identically satisfied we have complete integrability which is equivalent to constant curvature of Q , otherwise W must satisfy all equations (22) and (25). For example, when Q has dimension two, we have from (25)

$$(R_{1212} - mc \det(g_{ij})) \nabla^1 \ln W = 0,$$

and

$$(R_{2121} - mc \det(g_{ij})) \nabla^2 \ln W = 0.$$

Therefore, because of the symmetries of the Riemann tensor, we have

Theorem 6. *If Q has dimension 2, then equations (22) admit non-constant solutions W only if Q has constant curvature.*

For each l , the integrability conditions of (21) are weaker than those for the Hessian equation for $G(q^i)$ (3) and therefore the curvature of Q could be non-constant.

We give two examples in order to illustrate the Proposition 5.

Example 1. As shown in [5] and recalled above, when Q has constant curvature, equation (3), or equivalently equation (22), admits a solution depending on $n + 1$ real parameters (a_i) . Let G_i be a solution determined by the choice of a particular set of the (a_i) , let us assume that $G_i \neq G_j$. It is then natural to consider the relations between $U^m G_i$ and $U^m G_j$ and see if some choice of the parameters can provide new independent first integrals of the system. For example, let L be the natural Hamiltonian on the constant curvature manifold $Q = \mathbb{S}^2$ with $(q^1 = \theta, q^2 = \phi)$

$$L = \frac{1}{2}(p_\theta^2 + \frac{1}{\sin^2 \theta} p_\phi^2) + V. \quad (26)$$

A complete solution of a 0th degree $G(\theta, \phi, a_1, a_2, a_3)$ has been computed in [5]

$$G = (a_1 \sin \phi + a_2 \cos \phi) \sin \theta + a_3 \cos \theta. \quad (27)$$

and for $a_3 = 0$, the integration of equation (4) – or equivalently (24) – gives

$$V = \frac{1}{\cos^2 \theta} F((a_1 \sin \phi - a_2 \cos \phi) \tan \theta).$$

For different sets of the parameters (a_k) , $U^m G_i$ and $U^m G_j$ are no longer simultaneously first integrals of H unless if $F = F_0 = \text{constant}$, and therefore

$$V = \frac{F_0}{\cos^2 \theta}. \quad (28)$$

In this case, let be $G_1 = G(a_1 = 1, a_2 = 0, a_3 = 0)$ and $G_2 = G(a_1 = 0, a_2 = 1, a_3 = 0)$. Hence, for any extension of L of the form (1) with α given by (8), the five functions $L_0 = L$, $L_1 = p_2 = p_\phi$, H , $U^m G_1$ and $U^m G_2$ are functionally independent first integrals of H . For $m = 2$, recalling that $c = K/m$, the curvature of $Q = \mathbb{S}^2$ is $K = 1$ and choosing for the other free parameters of α the values $\kappa = 0$ and $u_0 = 0$, we have $\alpha = \frac{4}{u^2}$, and $U^2 G_1$ and $U^2 G_2$ are

$$\begin{aligned} U^2 G_1 &= (\sin \phi \sin \theta) \left(p_u^2 - p_\theta^2 \frac{4}{u^2} - F_0 \frac{8}{u^2 \cos^2 \theta} \right) + p_\theta p_u \frac{4}{u} \cos \theta \sin \phi \\ &+ p_\phi p_u \frac{4 \cos \phi}{u \sin \theta} - p_\phi^2 \frac{4 \sin \phi}{u^2 \sin \theta}, \\ U^2 G_2 &= \frac{(\cos^4 \theta + \sin^2 \theta - \cos^2 \theta) \cos \phi}{\sin^3 \theta} \left(p_u^2 - p_\theta^2 \frac{4}{u^2} - F_0 \frac{8}{u^2 \cos^2 \theta} \right) \\ &+ p_\theta p_u \frac{4}{u} \cos \theta \cos \phi - p_\phi p_u \frac{4 \sin \phi}{u \sin \theta} - p_\phi^2 \frac{4 \cos \phi}{u^2 \sin \theta}. \end{aligned}$$

Example 2. We can use a complete solution $G(q^i, a_k)$ of (3) in order to construct solutions λ^i of (21). Namely, we can choose $\lambda^i = G_i$, $i = 1, \dots, n$ where G_i denotes any particular solution of (3). We remark that it is not necessary that $G_i \neq G_j$ for $i \neq j$, or $G_i \neq 0$ for all i . By substituting the λ^i into (23), the equations become n second-order PDE in V whose solutions provide examples of compatible potentials. For instance, let us consider again L given by (26) on $Q = \mathbb{S}^2$. We can choose for λ^i the particular values $\lambda^1 = \cos(\theta)$, $\lambda^2 = 0$ of (27) as coefficients for a linear homogeneous G . Then, equations (23) can be integrated yielding,

$$V = \frac{c_1 + c_2 \sin \theta}{\cos^2 \theta},$$

which, for $c_2 \neq 0$ does not satisfies (4) with G given by (27), hence, for this potential the construction of $U^m G$ is possible only when G depends on the momenta. For the different choice of λ^i , $\lambda^1 = 0$, $\lambda^2 = \cos \theta$, the integration of (23) gives

$$V = \frac{c_1}{\sin^2 \theta},$$

which is compatible with G given by (27) for $a_1 = a_2 = 0$. The expressions of $U^m G$ can be computed by using (17).

Remark 5. By considering the functions λ^i as the components of a vector field Λ , equation (21) can be written as

$$[\mathbf{g}, [\mathbf{g}, \Lambda]] = -mc\Lambda \odot \mathbf{g}, \quad (29)$$

where $[\cdot, \cdot]$ are the Schouten-Nijenhuis brackets and \odot denotes symmetrized tensor product. This means that $[\mathbf{g}, \Lambda]$ is a particular kind of conformal Killing tensor, or, equivalently, that Λ is a particular conformal master symmetry of the geodesic equations, where the conformal factor is a constant multiple of Λ , instead of an arbitrary vector field. In a similar way, for G polynomial in the momenta of degree k with highest degree term given by $\lambda^{i_1 \dots i_k} p_{i_1} \dots p_{i_k}$, it is straightforward to show that a necessary condition for $U^m G$ to be first integral of H is still of the form (29), where now Λ is a k -tensor field. In the 0-th degree case $G = W(q^i)$ eq. (29) becomes $[\mathbf{g}, \nabla W] = -mcW \mathbf{g}$.

Definition 1. We call self-conformal (s-conformal in short) the $(k + 1)$ -order conformal Killing tensor field $[\mathbf{g}, \Lambda]$ such that

$$[\mathbf{g}, [\mathbf{g}, \Lambda]] = C \mathbf{g} \odot \Lambda,$$

$C \in \mathbb{R}$, is satisfied. In this case, the k -order tensor Λ is said to be a s-conformal master symmetry of the geodesic equations of \mathbf{g} .

In the case of $C = 0$, i.e. $c = 0$, s-conformal Killing tensors and master symmetries become the usual Killing tensors and master symmetries.

Theorem 7. Let G be a k -degree polynomial of degree k in the momenta. A necessary condition for $U^m G$ to be first integral of H is that the tensor Λ of components $\lambda^{i_1 \dots i_k}$ given by the coefficients of the highest-degree term of G is a self-conformal master symmetry of the geodesic equations of \mathbf{g} or, equivalently, that $[\mathbf{g}, \Lambda]$ is a self-conformal tensor field of \mathbf{g} with $C = -mc$.

The existence of a complete solution introduced in [5] and recalled above can be restated as follows

Corollary 8. Equation (3) admits a complete solution $G = W(q^i)$ if and only if the dimension of the space of the s-conformal Killing vectors ∇G , with $C = -mc$, is maximal and equal to $n + 1$.

4 Intrinsic characterisation of the extended Hamiltonians

We show under which geometrical conditions a $(n + 1)$ -dimensional natural Hamiltonian can be written as the extension (15) of a natural Hamiltonian L . Let us consider a natural Hamiltonian

$$H = \frac{1}{2} \tilde{g}^{ab} p_a p_b + \tilde{V} \quad (30)$$

on a $(n + 1)$ -dimensional Riemannian manifold $(\tilde{Q}, \tilde{\mathbf{g}})$ and let X be a conformal Killing vector of $\tilde{\mathbf{g}}$, that is a vector field satisfying

$$[X, \tilde{\mathbf{g}}] = \mathcal{L}_X \tilde{\mathbf{g}} = \phi \tilde{\mathbf{g}},$$

where ϕ is a function on \tilde{Q} and $[\cdot, \cdot]$ are the Schouten-Nijenhuis brackets. We denote by X^b the corresponding 1-form obtained by lowering the indices by means of the metric tensor $\tilde{\mathbf{g}}$.

Theorem 9. If on \tilde{Q} there exists a conformal Killing vector field X with conformal factor ϕ such that

$$dX^b \wedge X^b = 0, \quad (31)$$

$$d\phi \wedge X^b = 0, \quad (32)$$

$$d\|X\| \wedge X^b = 0, \quad (33)$$

$$X(\tilde{V}) = -\phi \tilde{V}, \quad (34)$$

$$\tilde{R}(X) = kX, \quad k \in \mathbb{R}, \quad (35)$$

where \tilde{R} is the Ricci tensor of the Riemannian manifold, then, there exist on \tilde{Q} coordinates (u, q^i) such that ∂_u coincides up to a rescaling with X and the natural Hamiltonian (30) has the form (15).

Proof. Condition (31) means that X is normal i.e., orthogonally integrable: locally there exists a foliation of n dimensional diffeomorphic manifolds Q such that $T_P Q = X^\perp = \{v \in T\tilde{Q} | \tilde{g}(v, X) = 0\}$; it follows that there exists a coordinate system $(q^0 = u, q^i)$ for $i = 1, \dots, n$ such that ∂_u is parallel to X and the components \tilde{g}^{0i} vanish for all $i = 1, \dots, n$. Furthermore, by (32) the conformal factor ϕ is constant on the leaves Q ($v(\phi) = 0$ for all $v \in X^\perp$); thus, ϕ depends only on u . By expanding the condition that $X = F(q^a)\partial_u$ is a conformal Killing vector

$$\left\{ \frac{1}{2}\tilde{g}^{00}(q^a)p_u^2 + \frac{1}{2}\tilde{g}^{ij}(q^a)p_i p_j, F(q^a)p_u \right\} = \phi(u) \left(\frac{1}{2}\tilde{g}^{00}(q^a)p_u^2 + \frac{1}{2}\tilde{g}^{ij}(q^a)p_i p_j \right)$$

we get the equations

$$\tilde{g}^{00}(2\partial_u F - \phi) - F\partial_u \tilde{g}^{00} = 0 \quad (36)$$

$$\tilde{g}^{hj}\partial_h F = 0 \quad j = 1, \dots, n \quad (37)$$

$$\tilde{g}^{ij}\phi + F\partial_u \tilde{g}^{ij} = 0 \quad i, j = 1, \dots, n \quad (38)$$

By (37), we have $F = F(u)$, hence due to (38) we get that $\partial_u \ln \tilde{g}^{ij}$ is a function of u , the same function for all i, j . Thus, without loss of generality we can assume $\tilde{g}^{ij} = g^{ij}(q^h)\alpha(u)$. Moreover, Eq. (36) implies that, up to a rescaling of u , \tilde{g}^{00} is independent of u . By imposing $X(\tilde{V}) = -\phi V$, we obtain $\partial_u \ln \tilde{V} = -\phi/F$, that means $\tilde{V} = \alpha(u)V(q^h)$, thus we get

$$H = \frac{1}{2}g^{00}(q^h)p_u^2 + \alpha(u) \left(\frac{1}{2}g^{ij}(q^h)p_i p_j + V(q^h) \right).$$

Finally, condition (33) means that the norm of X is constant on Q , that is $F(u)^2 g^{00}(q^i)$ is independent of (q^i) . This shows that up to a rescaling and a change of sign of H we can assume $g^{00} = 1$ and in the coordinate system (u, q^i) (30) has the required form (15). By computing again the Poisson bracket, we get relations between ϕ , α and F : $\alpha = k(F)^{-2}$ and $\phi = 2\dot{F}$ with k a real not vanishing constant. When X is a proper conformal Killing vector, we can assume that α is proportional to $F(u)^{-2}$. The covariant components of the Ricci tensor of \tilde{Q} are given in Lemma 10, in particular we have for $i = 1, \dots, n$

$$\tilde{R}_{00} = n \frac{\ddot{F}}{F}, \quad \tilde{R}_{0i} = 0.$$

Hence, $X = F(u)\partial_u$ is an eigenvector of the Ricci tensor with eigenvalue $\rho = n \frac{\ddot{F}}{F}$, which is constant if and only if F is proportional to $S_\kappa(cu + u_0)$. \square

Remark 6. If $\phi = 0$ (i.e., X is a Killing vector), then α and F are necessarily constant and this gives the geodesic term of the case $c = 0$, but equation (34) does not characterize the potential of the Hamiltonian (16).

Remark 7. It is straightforward to check that for a Hamiltonian of the form $H = \frac{1}{2}p_u^2 + F^{-2}(u)L$ with L a natural n -dimensional Hamiltonian, $X = F\partial_u$ is a CKV with conformal factor $\phi = 2\dot{F}$ such that $X(F^{-2}(u)V) = -\phi(F^{-2}(u)V)$. Hence, conditions of the above theorem are necessary for having an extended Hamiltonian of our form.

We want now to study the geometric properties of the metric $\tilde{\mathbf{g}}$ obtained by an extension of a metric \mathbf{g} , in particular when \mathbf{g} is of constant curvature.

In the following, we assume $\alpha(u) = f^{-2}$ in order to simplify computations. In particular, f is allowed to be pure imaginary.

Lemma 10. *Let (g_{ij}) be the components of a n -dimensional metric on Q in the coordinates (q^i) . We consider the $(n+1)$ -dimensional metric on \tilde{Q} having components \tilde{g}_{ab} ($a, b = 0, \dots, n$, $i, j = 1, \dots, n$) with respect to coordinates $(q^0 = u, q^i)$ defined as follows*

$$\tilde{g}_{ab} = \begin{cases} 1 & a = b = 0, \\ 0 & a = 0, b \neq 0, \\ f^2(u)g_{ij}(q^h) & a = i, b = j. \end{cases} \quad (39)$$

Then, the relations between the covariant components of the Riemann tensors associated with $\tilde{\mathbf{g}}$ and \mathbf{g} are for all $h, i, j, k = 1, \dots, n$

$$\tilde{R}_{h j k l} = f^2 R_{h j k l} - \frac{\dot{f}^2}{f^2} (\tilde{g}_{h k} \tilde{g}_{j l} - \tilde{g}_{h l} \tilde{g}_{j k}), \quad (40)$$

$$\tilde{R}_{0 j k l} = 0, \quad (41)$$

$$\tilde{R}_{0 j 0 l} = -\frac{\ddot{f}}{f} \tilde{g}_{j l}. \quad (42)$$

Moreover, the covariant components of the Ricci tensors R_{ij} and \tilde{R}_{ab} of the two metrics are related, for all $h, i, j, k = 1, \dots, n$, by

$$\tilde{R}_{00} = n \frac{\ddot{f}}{f}, \quad (43)$$

$$\tilde{R}_{0i} = 0, \quad (44)$$

$$\tilde{R}_{ij} = R_{ij} + \left(f \ddot{f} + (n-1) \dot{f}^2 \right) f^{-2} \tilde{g}_{ij}, \quad (45)$$

and the relation between the Ricci scalars R and \tilde{R} is

$$\tilde{R} = \frac{R}{f^2} + n \frac{2f \ddot{f} + (n-1) \dot{f}^2}{f^2}, \quad (46)$$

where \dot{f} and \ddot{f} denote the first and second derivative w.r.t. u of $f(u)$.

Expressions (40), (45), and (46) become simpler when Q is of constant curvature, while the other formulas remain unchanged.

Lemma 11. *Under the hypotheses of Lemma 10 with $n > 1$, if \mathbf{g} is a metric of constant curvature K , then the non zero covariant components of the Riemann tensor \tilde{R} associated with $\tilde{\mathbf{g}}$ are, for all $h, i, j, k = 1, \dots, n$*

$$\tilde{R}_{h j k l} = \frac{K - \dot{f}^2}{f^2} (\tilde{g}_{h k} \tilde{g}_{j l} - \tilde{g}_{h l} \tilde{g}_{j k}), \quad (47)$$

Moreover, the covariant components of the Ricci tensor \tilde{R}_{ij} and the Ricci scalar \tilde{R} are, for $i, j = 1, \dots, n$,

$$\tilde{R}_{ij} = \left(f \ddot{f} + (n-1)(\dot{f}^2 - K) \right) f^{-2} \tilde{g}_{ij}, \quad (48)$$

$$\tilde{R} = n \frac{2f \ddot{f} + (n-1)(\dot{f}^2 - K)}{f^2}. \quad (49)$$

Theorem 12. Let (Q, \mathbf{g}) be a n -dimensional Riemannian manifold of constant curvature $K = mc$ and $(\tilde{Q}, \tilde{\mathbf{g}})$ the extended manifold with metric (39), therefore

- i) the metric $\tilde{\mathbf{g}}$ is of constant curvature if and only if either $n = 1$ or $m = 1$ or $K = c = \dot{f} = 0$,
- ii) the metric $\tilde{\mathbf{g}}$ is conformally flat if and only if either $n > 2$ or $\tilde{\mathbf{g}}$ is of constant curvature.

Proof. For $n = 1$ the extended metric is, up to a rescaling of q^1 ,

$$\tilde{g}_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & f^2 \end{pmatrix},$$

which is of constant curvature if and only if \ddot{f} is proportional to f which is true if f is any trigonometric tagged function. For $n \geq 2$, due to the Bianchi identities, the metric is of constant curvature if the ratios

$$\tilde{R}_{abcd}/(\tilde{g}_{ac}\tilde{g}_{bd} - \tilde{g}_{ad}\tilde{g}_{bc})$$

are independent of (a, b, c, d) , that is by (47), (41), and (42)

$$\ddot{f}f + K - \dot{f}^2 = 0, \quad (50)$$

which for $c \neq 0$, i.e. $f^2 = \frac{S_\kappa(\frac{K}{m}u + u_0)}{K}$, becomes

$$\frac{K^2(m^2 - 1)}{m^2} = 0,$$

which holds only for $m = 1$ or for $K = c = 0$, when f is constant (see Theorem 2) and (50) holds.

For $n = 2$ the three-dimensional extended metric $\tilde{\mathbf{g}}$ is conformally flat if and only if the Weyl-Schouten tensor

$$\tilde{R}_{abc} = \tilde{\nabla}_c \tilde{R}_{ab} - \tilde{\nabla}_a \tilde{R}_{bc} + \frac{1}{2n} (\tilde{g}_{ac} \tilde{\nabla}_b \tilde{R} - \tilde{g}_{ab} \tilde{\nabla}_c \tilde{R}),$$

where $\tilde{\nabla}$ denotes the covariant derivative w.r.t. $\tilde{\mathbf{g}}$, vanishes. By applying the formulas derived in Lemma 11 we have that the only non vanishing components of \tilde{R}_{abc} are, for $i, k = 1, 2$,

$$\tilde{R}_{i0k} = \frac{\dot{f}}{f^3} \tilde{g}_{ik} (\ddot{f}f + K - \dot{f}^2),$$

which, as shown above, vanish only for $m = 1$ or in the case when $0 = K = c$ and f is constant. For $n > 2$ the $(n + 1)$ -dimensional extended metric $\tilde{\mathbf{g}}$ is conformally flat if and only if the Weyl tensor

$$\begin{aligned} \bar{C}_{abcd} &= \tilde{R}_{abcd} + \frac{1}{n-1} (\tilde{g}_{ac} \tilde{R}_{bd} - \tilde{g}_{ad} \tilde{R}_{bc} + \tilde{g}_{bd} \tilde{R}_{ac} - \tilde{g}_{bc} \tilde{R}_{ad}) + \\ &+ \frac{\tilde{R}}{n(n-1)} (\tilde{g}_{ad} \tilde{g}_{bc} - \tilde{g}_{ac} \tilde{g}_{bd}). \end{aligned}$$

vanishes and, by applying Lemma 11, this is true for all manifold Q of constant curvature. \square

5 Quantization

We consider here quantization for the case $m \leq 2$ only. For $m = 1$, it is well known how to associate a first order symmetry operator with any constant of motion linear in the momenta. In [2] the quantization of quadratic in the momenta first integrals of natural Hamiltonian functions has been analyzed and we recall here the results relevant for our case.

Let \hat{H} be the Hamiltonian operator associated with the Hamiltonian $H = \frac{1}{2}g^{ij}p_i p_j + V$, we have

$$\hat{H} = -\frac{\hbar^2}{2}\nabla_i(g^{ij}\nabla_j) + V = -\frac{\hbar^2}{2}\Delta + V,$$

where Δ is the Laplace-Beltrami operator. Let $T = \frac{1}{2}T^{ij}p_i p_j + V_T$ be a first integral of H , let

$$\hat{T} = -\frac{\hbar^2}{2}\nabla_i(T^{ij}\nabla_j) + V_T. \quad (51)$$

We have (Proposition 2.5 of [2])

Proposition 13. *Let be $\{H, T\} = 0$, then $[\hat{H}, \hat{T}] = 0$ if and only if*

$$\delta C = \delta(TR - RT) = 0, \quad (52)$$

where R is the Ricci tensor, T and R are considered as endomorphisms on vectors and one-forms and

$$(\delta A)^{ij\dots k} = \nabla_r A^{rij\dots k},$$

is the divergence operator for skew-symmetric tensor fields A .

For our purposes we need to apply (52) to the Ricci tensor of the extended metric and to the constant of the motion $T = U^2 G$. By assuming constant the curvature K of Q , the components of \tilde{R}_{ab} are given by inserting $f^2 = \frac{1}{K}S_\kappa^2(\frac{K}{m}u + u_0)$ or $f^2 = \frac{1}{mA}$ in Lemmas 10 and 11; the covariant components of the Ricci tensor are given respectively by

$$\begin{aligned} \tilde{R}_{00} &= -n \frac{\kappa K^2}{m^2}, \\ \tilde{R}_{0i} &= 0, \\ \tilde{R}_{ij} &= \frac{K^2}{m^2} \left(n\kappa + \frac{(n-1)(m^2-1)}{(T_k(\frac{K}{m}u + u_0))^2} \right) g^{ij}, \end{aligned}$$

for $K \neq 0$ and $\tilde{R}_{ab} = 0$ for $K = 0$.

In order to make computations easier, we remark that for A, B two-tensors on a Riemannian manifold $(\tilde{Q}, \tilde{\mathbf{g}})$ we have

$$(AB - BA)_c^a = A_b^a B_c^b - B_b^a A_c^b = A^{ad} B_{cd} - g^{ad} g_{ec} B_{db} A^{be}. \quad (53)$$

Lemma 14. *For any symmetric tensor T^{ij} the (1,1) components of $C = T\tilde{R} - \tilde{R}T$, where \tilde{R} is the Ricci tensor of $\tilde{\mathbf{g}}$, are*

$$\begin{aligned} C_0^0 &= 0, \\ C_0^i &= T^{0i}W, \\ C_i^0 &= -\tilde{g}_{ij}T^{0j}W, \\ C_j^i &= 0, \end{aligned}$$

where

$$W = (n-1) \frac{\ddot{f}f - \dot{f}^2 + K}{f^2}. \quad (54)$$

Remark 8. We immediately have that if $W = 0$ then $C = 0$ and, by Proposition 13, $\{H, T\} = 0$ implies $[\hat{H}, \hat{T}] = 0$. However, by Theorem 12, $W = 0$ if and only if either $n = 1$, $m = 1$ or f is constant, i.e., if and only if $\tilde{\mathbf{g}}$ is of constant curvature.

Theorem 15. For $m = 2$, $\{H, T\} = 0$ implies $[\hat{H}, \hat{T}] = 0$ if and only if $\tilde{\mathbf{g}}$ is of constant curvature i.e., if and only if $n = 1$ or f is constant.

Proof. If $K = 0$, and therefore $c = 0$ and f is constant, then $W = 0$. Otherwise, when $K \neq 0$ and $c \neq 0$, by computing $T = U^2 G$ and by applying Proposition 3 we get

$$\begin{aligned} T^{00} &= G, \\ T^{0i} &= \gamma \nabla^i G, \\ T^{ij} &= -\frac{K}{2} \gamma^2 G g^{ij}, \end{aligned}$$

where γ is given by

$$\gamma = (T_\kappa(\frac{K}{m}u + u_0))^{-1},$$

as proved in Theorem 2. A straightforward computation gives

$$\begin{aligned} \delta C_0 &= \gamma W (g^{il} \partial_{il}^2 G + \partial_l G (\partial_i g^{il} + g^{il} \partial_i \ln \sqrt{g})) = \\ &= \gamma W \Delta G = -\gamma n K W G, \\ \delta C_i &= f \partial_i G \frac{d}{du} (\gamma f W), \end{aligned}$$

where $g = \det(g_{ij})$. By inserting the expressions of γ and of $f^2 = \frac{S_\kappa^2(\frac{K}{m}u + u_0)}{K}$ we have that there are no non-trivial ($G \neq \text{const.}$) solutions to $\delta C = 0$ other than those such that $W = 0$, that is, after Remark 8, when $n = 1$ or \tilde{Q} is of constant curvature. \square

In a recent paper [1], where particular conformally flat, non-constant curvature manifolds are considered, it is shown that even if the Laplace-Beltrami quantization of some first integrals of the Hamiltonian fails, their quantization is somehow made possible by considering the conformal Schrödinger operator instead of the standard (Laplace-Beltrami) one. The conformal Schrödinger operator is related to the standard one by an additional term proportional to the scalar curvature. In Theorem 12, we proved that our extended Hamiltonians for $n > 2$ have always conformally flat configuration manifolds, therefore, the method exposed in [1] could be, at least in principle, applicable.

If we denote by $\tilde{\Delta}$ the Laplace-Beltrami operator of $(\tilde{Q}, \tilde{\mathbf{g}})$ and by Δ the Laplace-Beltrami operator of the constant curvature manifold (Q, \mathbf{g}) , a direct calculation shows that

$$\tilde{\Delta} = \partial_u^2 + n \frac{\dot{f}}{f} \partial_u + \frac{K}{f^2} \Delta, \quad (55)$$

and $[\tilde{\Delta}, \Delta] = 0$. Therefore, being

$$\hat{H} = -\frac{\hbar^2}{2} (\partial_u^2 + n \frac{\dot{f}}{f} \partial_u) + \frac{K}{f^2} \hat{L},$$

with

$$\hat{L} = -\frac{\hbar^2}{2}\Delta + V,$$

we have

Proposition 16. *\hat{L} is a symmetry operator of \hat{H} :*

$$[\hat{H}, \hat{L}] = 0.$$

Since \hat{H} and \hat{L} have common eigenfunctions, from $\hat{H}\psi = \mu\psi$ and $\hat{L}\psi = \lambda\psi$ we obtain for the eigenfunction of \hat{H} the following characterization

Proposition 17. *The function $\psi(u, q^i)$ is an eigenfunction of \hat{H} if and only if ψ is an eigenfunction of \hat{L} and*

$$-\frac{\hbar^2}{2}(\partial_u^2\psi + n\frac{\dot{f}}{f}\partial_u\psi) + \left(\frac{K\lambda}{f^2} - \mu\right)\psi = 0. \quad (56)$$

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References

- [1] Ballesteros A, Enciso A, Herranz F J, Ragnisco O and Riglioni D 2011 Quantum mechanics on spaces of nonconstant curvature: The oscillator problem and superintegrability *Ann. Phys.* **326** n.8, 2053-2073
- [2] Benenti S, Chanu C and Rastelli G 2002 Remarks on the connection between the additive separation of the Hamilton-Jacobi equation and the multiplicative separation of the Schrödinger equation. II. First integrals and symmetry operators *J. Math. Phys.* **43** 5223-5253
- [3] Cariñena J F, Rañada M F and Santander M 2005 Central potentials on spaces of constant curvature: The Kepler problem on the two-dimensional sphere S^2 and the hyperbolic plane H^2 *J. Math. Phys.* **46** no. 5, 052702
- [4] Chanu C, Degiovanni L and Rastelli G 2008 Superintegrable three-body systems on the line, *J. Math. Phys.* **49**, 112901
- [5] Chanu C, Degiovanni L, Rastelli G 2011 First integrals of extended Hamiltonians in $(n+1)$ -dimensions generated by powers of an operator, *SIGMA* **7** 038, 12 pages
- [6] Chanu C, Degiovanni L and Rastelli G 2011 Polynomial constants of motion for Calogero-type systems in three dimensions *J. Math. Phys.* **52** 032903
- [7] Kalnins E G, Kress J M and Miller W Jr 2010 Tools for Verifying Classical and Quantum Superintegrability *SIGMA* **6** 066
- [8] Maciejewski A J, Przybylska M and Yoshida H 2010 Necessary conditions for super-integrability of a certain family of potentials in constant curvature spaces, *J. Phys. A: Math. Theor.* **43** 382001

- [9] Rañada M F and Santander M 1999 Superintegrable systems on the two-dimensional sphere S^2 and the hyperbolic plane H^2 *J. Math. Phys.* **40** 5026-5057
- [10] Tremblay F, Turbiner A V and Winternitz P 2009 An infinite family of solvable and integrable quantum systems on a plane *J. Phys. A* **42** n. 24, 242001, 10 pp.